HOMOGENEITY AND CURVATURES OF GEODESIC SPHERES

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Dedicated to Professor Dmitri V. Alekseevsky on the occasion of his 65th birthday

ABSTRACT. We study some scalar curvature invariants on geodesic spheres and use them to characterize several kinds of Riemannian manifolds such as homogenous manifolds and in particular, the two-point homogeneous spaces and the Damek-Ricci spaces.

1. INTRODUCTION

In order to understand the geometry of a Riemannian manifold different objects have been considered. Among them, the symmetries of the manifold, bundles where the manifold is the base or families of submanifolds such as geodesic spheres, tubes or disks, have been used extensively [10], [11], [13], [14]. The properties of those objects influence and are influenced by the geometry of the ambient space (see, for example, [5], [16], and the references therein).

On the other hand, the fact that the curvature tensor of a Riemannian manifold is so difficult to handle, motivated the investigation of functions or operators associated to it. Examples are the sectional curvature, the scalar curvature and the Jacobi operator, among others [2], [8]. In this paper, we focus on scalar curvature invariants which reflect important properties, such as local homogeneity [15].

The purpose of this work is to link the study of geodesic spheres with the investigation of scalar curvature invariants [3], [4], [5]. The whole space of scalar curvature invariants is generated by the so-called Weyl invariants and this suggested to consider only Weyl invariants at first glance. In view of our applications, the simple ones, that is, those not involving covariant derivatives of the curvature tensor are the ones to start with.

For an arbitrary simple Weyl invariant on a geodesic sphere, we give an explicit expression of the first terms in its power series expansion in function of the radius, as shown in Theorem 3.4. By integrating that invariant, we obtain the corresponding total scalar curvature of a geodesic sphere, whose power series expansion is discussed in Theorem 4.4. The geometrical meaning of the first terms in those expansions is studied, leading to characterizations of the two-point homogeneous spaces among Riemannian manifolds with adapted holonomy (Theorem 5.6). We emphasize that unlike the volume conjecture [11], where such a characterization has not been achieved in the most general context, it can be obtained for a number of total scalar curvatures (Example 5.7).

1991 *Mathematics Subject Classification.* 53C25, 53C30.

Key words and phrases. Curvature invariants, Damek-Ricci spaces, geodesic spheres, homogeneous spaces, two-point homogeneous spaces.

Supported by Project BFM 2003-02949 (Spain).

The paper is organized as follows. In Section 2, we introduce the notions and conventions used throughout the paper. In Section 3, we give power series expansions for several geometric objects defined on geodesic spheres and use them to give some first characterizations for some classes of spaces such as Damek-Ricci and two-point homogeneous spaces. Section 4 deals with total scalar curvature of geodesic spheres. Finally, we give further characterization results for homogeneous and two-point homogeneous manifolds in Section 5.

2. Curvature and Weyl invariants

Let (M^n, g) be a Riemannian manifold of dimension n. We denote by ∇ the Levi Civita connection of g, and by R its curvature tensor defined by $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, where X and Y are vector fields on M. We set $R_{XYVW} = g(R_{XY}V, W)$. The Ricci tensor is defined by $\rho = \operatorname{tr}_{13} R$, where tr_{13} means the trace with respect to the first and third indices. For any tensor field ω we set $\nabla^0\omega := \omega$ and we denote by $\nabla^l\omega$ the l-the covariant derivative of ω.

We now explain some notions from the theory of invariants, thereby mainly following [15]. Let $\mathcal{F}M = (\mathcal{F}M, \pi, M, Gl(n, \mathbb{R}))$ be the bundle of linear frames over (M, g). For $k \geq 1$ we shall denote by $T^kM = \cup_{m\in M}(T_mM \times \cdots \times T_mM)$ the bundle over M with standard fibre $\mathbb{R}^n \times \overset{k}{\ldots} \times \mathbb{R}^n$ and structure group $Gl(n, \mathbb{R})$ which is associated to the principal bundle $\mathcal{F}M$. If $k = 0$ we set $T^0M := M$.

A *partial Weyl invariant*, W, with k degrees of freedom is a map

$$\begin{array}{llll} W: & T^kM & \longrightarrow & \mathbb{R} \\ & (v_1, \ldots, v_k) & \mapsto & \operatorname{tr}(g \otimes \cdots \otimes g \otimes \nabla^{l_1} R \otimes \cdots \otimes \nabla^{l_\nu} R)(v_1, \ldots, v_k) \end{array} \tag{2.1}$$

where $l_j \in \mathbb{N} \cup \{0\}$, $j \in \{1, \ldots, \nu\}$, $\nu \in \mathbb{N}$, and tr is a product of traces [1] with respect to some permutation of the indices. Two partial Weyl invariants W_1 and W_2 are equal if and only if $W_1(v_1, \ldots, v_k) = W_2(v_1, \ldots, v_k)$ for any $(v_1, \ldots, v_k) \in T^kM$ and every Riemannian manifold (M, g).

We say that a partial Weyl invariant W is *simple* if its construction does not involve covariant derivatives of the curvature tensor, that is, W can be written as $W = \operatorname{tr}(g \otimes \cdots \otimes g \otimes R \otimes \cdots \otimes R)$.

In particular, a *Weyl invariant*, as defined in [15], is a partial Weyl invariant with zero degrees of freedom, that is, $k = 0$.

We define the *degree* of a partial Weyl invariant given by (2.1) as

$$\deg W = l_1 + \cdots + l_\nu + 2\nu. \tag{2.2}$$

We point out that other authors define the degree (or order) of a curvature invariant as half this number. Equivalently, the degree of a partial Weyl invariant is half the number of derivatives of the metric tensor involved in its construction. Clearly, if W_1 and W_2 are two partial Weyl invariants, then W_1W_2 can be considered as another partial Weyl invariant in the obvious way and

$$\deg W_1W_2 = \deg W_1 + \deg W_2. \tag{2.3}$$

For instance, the curvature tensor R and the Ricci tensor ρ are simple partial Weyl invariants of degree 2, the former with 4 degrees of freedom and the latter with 2.

By definition, a *partial scalar curvature invariant* is a linear combination of partial Weyl invariants. If all partial Weyl invariants involved in the construction of a partial scalar curvature invariant have the same degree d, then this partial scalar curvature invariant is said to have degree d.

Given a tangent vector $u \in TM$ and a partial scalar curvature invariant S with k degrees of freedom, we say that $S(u, \overset{k}{\ldots}, u)$ is a *partial directional curvature invariant* or to be more specific, a *partial curvature invariant in the direction of* u.

A scalar curvature invariant, as defined in [11] or [15], is a polynomial in the components of the curvature tensor and its covariant derivatives that does not depend on the choice of orthonormal basis used to build it. It follows from Weyl's theory of invariants [1], [17], that the scalar curvature invariants are precisely the traces of the curvature tensor and its covariant derivatives. As a consequence, a *scalar curvature invariant* is a linear combination of Weyl invariants. Alternatively, a scalar curvature invariant is a partial scalar curvature invariant with zero degrees of freedom. We recall here that a scalar curvature invariant has always even degree.

Let us denote by $I(\nu, n)$ the vector space of simple scalar curvature invariants of degree 2ν in a manifold of dimension n.

It is well known that for $n \geq 2$, $I(1, n)$ is a vector space of dimension 1 generated by the scalar curvature $\tau = \operatorname{tr} \rho$, that is, $\tau = \sum_{i=1}^{n} \rho_{ii}$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_mM. From now on, we use the notation $\omega_{ijk\ldots} = \omega(e_i, e_j, e_k, \ldots)$, $\nabla^{\alpha}_{ijk\ldots} = \nabla^{\alpha}_{e_i, e_j, e_k, \ldots}$, and so on.

If $n \geq 4$, $I(2, n)$ is a vector space of dimension 3 spanned by

$$\tau^2, \quad \|R\|^2 = \sum R^2_{ijkl}, \quad \|\rho\|^2 = \sum \rho^2_{ij}. \tag{2.4}$$

For $n \geq 6$, the vector space $I(3, n)$ has dimension 8 and it is spanned by the following basis:

$$\begin{array}{llll}
\tau^3, & & \langle \rho \otimes \rho, \bar{R} \rangle & = \sum \rho_{ij} \rho_{kl} R_{ikjl}, \\
\tau \|\rho\|^2, & & \langle \rho, \dot{R} \rangle & = \sum \rho_{ij} R_{iklp} R_{jklp}, \\
\tau \|R\|^2, & & \breve{R} & = \sum R_{ijkl} R_{ijpq} R_{klpq}, \\
\breve{\rho} & = \sum \rho_{ij} \rho_{ik} \rho_{jk}, & \breve{\breve{R}} & = \sum R_{ijkl} R_{ipkq} R_{jplq}.
\end{array} \tag{2.5}$$

Remark 2.1. Let $W = \operatorname{tr}(R \otimes \overset{\nu}{\cdots} \otimes R)$ be a Weyl invariant of degree 2ν. In an $(n-1)$-dimensional Riemannian manifold of constant sectional curvature λ the curvature tensor can be written as $R = \lambda R^0$, where R^0 can be expressed with respect to an orthonormal basis as

$$R^0_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{2.6}$$

Then, $W = \operatorname{tr}(R \otimes \overset{\nu}{\cdots} \otimes R) = \lambda^\nu \operatorname{tr}(R^0 \otimes \overset{\nu}{\cdots} \otimes R^0) = \bar{A}_W(n-1)\lambda^\nu$ where $\bar{A}_W$ is a polynomial that only depends on W. Moreover, if $n \in \{1, 2\}$, then $R = 0$ and hence we have $\bar{A}_W(0) = \bar{A}_W(1) = 0$. Thus, $\bar{A}_W$ can be written as $\bar{A}_W(n-1) = (n-1)(n-2)A_W(n-1)$, where A_W is another polynomial. Then, for the constant curvature case we have

$$W = (n-1)(n-2)A_W(n-1)\lambda^\nu. \tag{2.7}$$

Example 2.2. The polynomials A_W corresponding to the Weyl invariants (2.4) and (2.5) can be explicitly given as follows. Suppose (M^{n-1}, g) has constant sectional curvature λ. First, we have $\tau = (n-1)(n-2)\lambda$, and thus,

$$A_\tau(n-1) = 1. \tag{2.8}$$

Also, for the Weyl invariants of degree 4,

$$A_{\|R\|^2}(n-1) = 2, \quad A_{\|\rho\|^2}(n-1) = n-2, \quad A_{\tau^2}(n-1) = (n-1)(n-2). \tag{2.9}$$

The expressions corresponding to the invariants (2.5) are summarized in the following table:

W	$A_W(n-1)$	W	$A_W(n-1)$
τ^3	$(n-1)^2(n-2)^2$	$\langle\rho\otimes\rho,\bar{R}\rangle$	$(n-2)^2$
$\tau\|\rho\|^2$	$(n-1)(n-2)^2$	$\langle\rho,\dot{R}\rangle$	$2(n-2)$
$\tau\|R\|^2$	$2(n-1)(n-2)$	$\bar{R}$	4
$\check{\rho}$	$(n-2)^2$	$\check{\bar{R}}$	$n-3$

3. Weyl invariants for geodesic spheres

A *geodesic sphere*, $G_m(r)$, with center m and radius r, is defined as

$$G_m(r) = \exp_m\left(S^{n-1}(r)\right), \tag{3.1}$$

where $\exp_m$ is the exponential map at m and $S^{n-1}(r)$ is the Euclidean sphere of radius r in the tangent space at m, that is, $S^{n-1}(r) = \{x \in T_mM : g(x,x) = r^2\}$. We always assume that $r < i(m)$, where $i(m)$ is the injectivity radius at the point m. Hence, geodesic spheres are the level sets of the radial distance function, that is, $G_m(r) = \{p \in M : d(m,p) = r\}$.

The purpose of this section is to study simple Weyl invariants on geodesic spheres. Thus, it would be convenient to express them in terms of geometrical data of the ambient manifold. Unfortunately, an explicit expression has not been achieved so far in full generality, not even in the simplest cases. We will content ourselves with the power series expansion of a simple Weyl invariant of a geodesic sphere. That will be enough for the purposes of this paper. In order to provide the power series expansion of a Weyl invariant, we use the second fundamental form of a geodesic sphere. Then, its curvature tensor can be calculated from the Gauss equation and the expression of an arbitrary Weyl invariant can then be obtained from it.

Lemma 3.1. *If σ denotes the second fundamental form of the geodesic sphere $G_m(r)$, then we have the power series expansion*

$$\sigma_{ij}\left(\exp_m(ru)\right) = \sum_{\alpha=-1}^{s-1} \frac{r^\alpha}{(\alpha+1)!}\,\sigma_{ij}^{\alpha+1}(u) + O\left(r^s\right), \tag{3.2}$$

where $\sigma_{ij}^{\alpha}(u)$, $\alpha \geq 2$, is a partial scalar curvature invariant of M at m with $\alpha+2$ degrees of freedom and degree α. The first terms of this expansion are

$$\begin{aligned}\sigma_{ij}\left(\exp_m(ru)\right) =& \frac{1}{r}\delta_{ij} - \frac{r}{3}R_{uiuj}(m) - \frac{r^2}{4}\nabla_u R_{uiuj}(m)\\ &-r^3\left(\frac{1}{45}\sum_{a=1}^{n} R_{uiua}R_{ujua} + \frac{1}{10}\nabla^2_{uu}R_{uiuj}\right)(m) + O\left(r^4\right).\end{aligned} \tag{3.3}$$

Proof. Using the Ledger recursion formula [4], [16], we have

$$\sigma_{ij}^0(u) = \delta_{ij}, \quad \sigma_{ij}^1(u) = 0,$$
$$\sigma_{ij}^\alpha(u) = -\frac{\alpha(\alpha-1)}{\alpha+1}\nabla^{\alpha-2}_{u\ldots u}R_{uiuj}(m) - \frac{1}{\alpha+1}\sum_{\beta=2}^{\alpha-2}\binom{\alpha}{\beta}\sum_{\gamma=1}^{n}\sigma_{i\gamma}^{\beta}(u)\sigma_{\gamma j}^{\alpha-\beta}(u), \tag{3.4}$$

for $\alpha \geq 2$. The result now follows by induction. □

Lemma 3.2. *Let $\widetilde{R}$ denote the curvature tensor of a geodesic sphere $G_m(r)$. Then*

$$\widetilde{R}_{ijkl}\left(\exp_m(ru)\right) = \sum_{\alpha=-2}^{s-2} r^{\alpha} \widetilde{R}^{\alpha+2}_{ijkl}(u) + O\left(r^{s-1}\right), \tag{3.5}$$

where $\widetilde{R}^{\alpha}_{ijkl}(u)$, $\alpha \geq 2$, is a partial curvature invariant at m of degree α such that for all the Weyl invariants used in its construction the number of degrees of freedom has the same parity as α. More specifically we obtain

$$\begin{aligned}
\widetilde{R}_{ijkl}\left(\exp_m(ru)\right) &= \frac{1}{r^2}\Big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\Big) \\
&+\Big(R_{ijkl} - \frac{1}{3}\delta_{ik}R_{ujul} + \frac{1}{3}\delta_{il}R_{ujuk} + \frac{1}{3}\delta_{jk}R_{uiul} - \frac{1}{3}\delta_{jl}R_{uiuk}\Big)(m) \\
&+r\,\Big(\nabla_u R_{ijkl} - \frac{1}{4}\delta_{jl}\nabla_u R_{uiuk} + \frac{1}{4}\delta_{jk}\nabla_u R_{uiul} \\
&\qquad + \frac{1}{4}\delta_{il}\nabla_u R_{ujuk} - \frac{1}{4}\delta_{ik}\nabla_u R_{ujul}\Big)(m) \\
&+r^2\,\Big(-\frac{1}{9}R_{uiul}R_{ujuk} + \frac{1}{9}R_{uiuk}R_{ujul} \\
&\qquad - \frac{1}{45}\delta_{ik}\sum_{a=1}^{n} R_{ujua}R_{ulua} + \frac{1}{45}\delta_{il}\sum_{a=1}^{n} R_{ujua}R_{ukua} \\
&\qquad + \frac{1}{45}\delta_{jk}\sum_{a=1}^{n} R_{uiua}R_{ulua} - \frac{1}{45}\delta_{jl}\sum_{a=1}^{n} R_{uiua}R_{ukua} \\
&\qquad + \frac{1}{2}\nabla^2_{uu}R_{ijkl} - \frac{1}{10}\delta_{jl}\nabla^2_{uu}R_{uiuk} + \frac{1}{10}\delta_{jk}\nabla^2_{uu}R_{uiul} \\
&\qquad + \frac{1}{10}\delta_{il}\nabla^2_{uu}R_{ujuk} - \frac{1}{10}\delta_{ik}\nabla^2_{uu}R_{ujul}\Big)(m) + O\left(r^3\right).
\end{aligned} \tag{3.6}$$

Proof. The Gauss equation relates the intrinsic curvature tensor of a submanifold to that of the ambient space by means of the second fundamental form as follows:

$$\widetilde{R}_{xyvw} = R_{xyvw} + \sigma_{xv}\sigma_{yw} - \sigma_{xw}\sigma_{yv}. \tag{3.7}$$

Using the power series expansion along a geodesic with respect to a parallel basis $R_{ijkl}\left(\exp_m(ru)\right) = R_{ijkl}(m) + r\nabla_u R_{ijkl}(m) + \frac{r^2}{2}\nabla^2_{uu}R_{ijkl}(m) + \cdots$ and plugging the above expression and (3.2) into (3.7), we have [4], [5]

$$\begin{aligned}
\widetilde{R}^0_{ijkl}(u) &= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} = R^0_{ijkl}, \\
\widetilde{R}^1_{ijkl}(u) &= 0, \\
\widetilde{R}^{\alpha}_{ijkl}(u) &= \frac{1}{(\alpha-2)!}\nabla^{\alpha-2}_{u\ldots u}R_{ijkl}(m) \\
&\quad + \frac{1}{\alpha!}\sum_{\beta=1}^{\alpha-3}\binom{\alpha}{\beta+1}\Big(\sigma^{\beta+1}_{ik}(u)\sigma^{\alpha-\beta-1}_{jl}(u) - \sigma^{\beta+1}_{il}(u)\sigma^{\alpha-\beta-1}_{jk}(u)\Big),
\end{aligned} \tag{3.8}$$

for $\alpha \geq 2$, and the first part of the result follows by induction. Finally, in formula (3.8) there are two clearly different terms. The first one $\frac{1}{(\alpha-2)!}\nabla^{\alpha-2}_{u\cdots u}R_{ijkl}(m)$ is a partial scalar curvature invariant with $\alpha+2$ degrees of freedom. The second addend is another partial curvature invariant with $\alpha+4$ degrees of freedom. The last statement is clear from Lemma 3.1 and (2.3). □

The following lemma is a technical result that will be needed in Theorem 3.4.

Lemma 3.3. *Let $(V, \langle\, ,\, \rangle)$ be an inner product vector space of dimension $n \geq 2$ and* tr *a total trace in the space of covariant tensors of order 4ν over V. If R is an algebraic curvature tensor on V, $Sc(R)$ its scalar curvature and W the algebraic invariant defined by $W = tr(R \otimes \overset{\nu}{\cdots} \otimes R)$, then*

$$\operatorname{tr}\left(\sum_{\alpha=1}^{\nu} R^0 \otimes \cdots \otimes \overset{\overset{\alpha}{\downarrow}}{R} \otimes \cdots \otimes R^0\right) = \nu A_W(n) Sc(R). \tag{3.9}$$

Proof. Clearly, $\operatorname{tr}\left(\sum_{\alpha=1}^{\nu} R^0 \otimes \cdots \otimes R \otimes \cdots \otimes R^0\right)$ is a scalar curvature invariant of degree two, and hence it is a multiple of the scalar curvature. Then, put

$$a\, Sc(R) = \operatorname{tr}\left(\sum_{\alpha=1}^{\nu} R^0 \otimes \cdots \otimes \overset{\overset{\alpha}{\downarrow}}{R} \otimes \cdots \otimes R^0\right). \tag{3.10}$$

The above formula is true for each algebraic curvature tensor R in V. If we take $R = R^0$ we have

$$\begin{aligned} a\, n(n-1) &= \sum_{\alpha=1}^{\nu} \operatorname{tr}\left(R^0 \otimes \cdots \otimes \overset{\overset{\alpha}{\downarrow}}{R^0} \otimes \cdots \otimes R^0\right) \\ &= \nu \operatorname{tr}\left(R^0 \otimes \cdots \otimes R^0\right) = \nu\, n(n-1) A_W(n). \end{aligned} \tag{3.11}$$

Thus $a = \nu\, A_W(n)$. □

Theorem 3.4. *Let $\widetilde{W}$ be a simple intrinsic Weyl invariant of degree 2ν, $\nu > 1$, on a geodesic sphere $G_m(r)$. Then we have*

$$\widetilde{W}\left(\exp_m(ru)\right) = \sum_{\alpha=-2\nu}^{s-2\nu} r^{\alpha} \widetilde{W}_{\alpha+2\nu}(u) + O\left(r^{s-2\nu+1}\right), \tag{3.12}$$

where $\widetilde{W}_{\alpha}(u)$ is a partial curvature invariant of degree α in the direction of u such that the degree of freedom of all the Weyl invariants involved in its construction has the same parity as α. More specifically, we have

$$\begin{aligned} \widetilde{W}_0(u) &= (n-1)(n-2)A_W(n-1), \\ \widetilde{W}_1(u) &= 0, \\ \widetilde{W}_2(u) &= \nu\, A_W(n-1)\left(\tau - \frac{2(n+1)}{3}\rho_{uu}\right)(m), \\ \widetilde{W}_3(u) &= \nu\, A_W(n-1)\left(\nabla_u \tau - \frac{n+2}{2}\nabla_u \rho_{uu}\right)(m), \\ \widetilde{W}_4(u) &= \omega_4(u) + \frac{\nu}{2} A_W(n-1)\left(\nabla^2_{uu}\tau - \frac{2(n+3)}{5}\nabla^2_{uu}\rho_{uu}\right)(m), \end{aligned} \tag{3.13}$$

where $\omega_4(u)$ is a simple directional curvature invariant of degree four given by

$$
\begin{aligned}
\omega_4(u) \;=\;& \nu\, A_W(n-1)\left(-\frac{2n+1}{45}\sum_{a,b=1}^{n} R_{uaub}^2+\frac{1}{9}\rho_{uu}^2\right)(m)\\
&+B_W^1(n-1)\left(\|R\|^2-4\sum_{a,b,c=1}^{n} R_{uabc}^2+\frac{4(n+12)}{9}\sum_{a,b=1}^{n} R_{uaub}^2\right.\\
&\qquad\left.-\frac{8}{3}\sum_{a,b=1}^{n}\rho_{ab}R_{uaub}+\frac{4}{9}\rho_{uu}^2\right)(m)\\
&+B_W^2(n-1)\left(\|\rho\|^2+\frac{n^2}{9}\sum_{a,b=1}^{n} R_{uaub}^2-\frac{2n}{3}\sum_{a,b=1}^{n}\rho_{ab}R_{uaub}\right.\\
&\qquad\left.-2\sum_{a=1}^{n}\rho_{ua}^2+\frac{3n+14}{9}\rho_{uu}^2-\frac{2}{3}\tau\rho_{uu}\right)(m)\\
&+B_W^3(n-1)\left(\tau-\frac{2(n+1)}{3}\rho_{uu}\right)^2(m),
\end{aligned}
\tag{3.14}
$$

and B_W^1, B_W^2 and B_W^3 are polynomials depending only on the dimension $n-1$ and satisfying

$$
2\,B_W^1(n-1)+(n-2)B_W^2(n-1)+(n-1)(n-2)B_W^3(n-1)=\binom{\nu}{2}A_W(n-1). \tag{3.15}
$$

Moreover, B_W^1, B_W^2 and B_W^3 are defined by formula (3.14).

Proof. Using the notation of Lemma 3.2, we have

$$
\begin{aligned}
&(\widetilde{R}\otimes\cdots\otimes\widetilde{R})_{i_1j_1k_1l_1\cdots i_\nu j_\nu k_\nu l_\nu}=\\
&\quad\sum_{\alpha=-2\nu}^{s-2\nu} r^\alpha\left(\sum_{\beta_1+\cdots+\beta_\nu=\alpha}\widetilde{R}^{\beta_1+2}_{i_1j_1k_1l_1}(u)\cdots\widetilde{R}^{\beta_\nu+2}_{i_\nu j_\nu k_\nu l_\nu}(u)\right)+O\left(r^{s-2\nu+1}\right).
\end{aligned}
\tag{3.16}
$$

By taking traces in (3.16), the result of Lemma 3.2 and the rule (2.3) to compute the degrees, we get the first result of Theorem 3.4.

Now, we turn our attention to the explicit expressions (3.13). As $\widetilde{R}$ is the curvature tensor of the geodesic sphere $G_m(r)$, the coefficients of its power series expansion (3.5) are algebraic curvature tensors in $u^\perp$. Using (3.6) and Remark 2.1, we get

$$
\begin{aligned}
\widetilde{W_0}(u) &= \operatorname{tr}\left(\widetilde{R}^0_{i_1j_1k_1l_1}(u)\cdots\widetilde{R}^0_{i_\nu j_\nu k_\nu l_\nu}(u)\right)\\
&= \operatorname{tr}\left(R^0\otimes\overset{\nu}{\cdots}\otimes R^0\right)\\
&= (n-1)(n-2)A_W(n-1).
\end{aligned}
\tag{3.17}
$$

Using the notation of Lemma 3.2, as $\widetilde{R}^1=0$, we have

$$
\widetilde{W_1}(u)=\operatorname{tr}\left(\sum_{\alpha=1}^{\nu}\widetilde{R}^0_{i_1j_1k_1l_1}(u)\cdots\widetilde{R}^1_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots\widetilde{R}^0_{i_\nu j_\nu k_\nu l_\nu}(u)\right)=0. \tag{3.18}
$$

Using Lemma 3.3 and (3.6), we obtain

$$\begin{aligned}\widetilde{W}_2(u) &= \operatorname{tr}\left(\sum_{\alpha=1}^{\nu} \widetilde{R}^0_{i_1j_1k_1l_1}(u)\cdots\widetilde{R}^2_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots\widetilde{R}^0_{i_\nu j_\nu k_\nu l_\nu}(u)\right)\\ &= \operatorname{tr}\left(\sum_{\alpha=1}^{\nu} R^0_{i_1j_1k_1l_1}\cdots\widetilde{R}^2_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots R^0_{i_\nu j_\nu k_\nu l_\nu}\right)\\ &= \nu A_W(n-1)Sc(\widetilde{R}^2),\end{aligned} \tag{3.19}$$

and from the definition of $\widetilde{R}^2$ in (3.6) we have $Sc(\widetilde{R}^2)=\tau-\frac{2(n+1)}{3}\rho_{uu}$. Hence

$$\widetilde{W}_2(u)=\nu A_W(n-1)\left(\tau-\frac{2(n+1)}{3}\rho_{uu}\right)(m). \tag{3.20}$$

Similarly, using Lemma 3.3, (3.6) and the fact that $Sc(\widetilde{R}^3)=\nabla_u\tau-\frac{n+2}{2}\nabla_u\rho_{uu}$, we obtain

$$\begin{aligned}\widetilde{W}_3(u) &= \operatorname{tr}\left(\sum_{\alpha=1}^{\nu} \widetilde{R}^0_{i_1j_1k_1l_1}(u)\cdots\widetilde{R}^3_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots\widetilde{R}^0_{i_\nu j_\nu k_\nu l_\nu}(u)\right)\\ &= \operatorname{tr}\left(\sum_{\alpha=1}^{\nu} R^0_{i_1j_1k_1l_1}\cdots\widetilde{R}^3_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots R^0_{i_\nu j_\nu k_\nu l_\nu}\right)\\ &= \nu\, A_W(n-1)\left(\nabla_u\tau-\frac{n+2}{2}\nabla_u\rho_{uu}\right)(m).\end{aligned} \tag{3.21}$$

Finally, using (3.16), we get for $\nu>1$,

$$\begin{aligned}\widetilde{W}_4(u) = {}& \operatorname{tr}\left(\sum_{\alpha=1}^{\nu} R^0_{i_1j_1k_1l_1}\cdots\widetilde{R}^4_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots R^0_{i_\nu j_\nu k_\nu l_\nu}\right)\\ &+\operatorname{tr}\Bigg(\sum_{\alpha<\beta} R^0_{i_1j_1k_1l_1}\cdots\widetilde{R}^2_{i_\alpha j_\alpha k_\alpha l_\alpha}(u)\cdots\widetilde{R}^0_{i_\gamma j_\gamma k_\gamma l_\gamma}(u)\cdots\\ &\qquad\cdots\widetilde{R}^2_{i_\beta j_\beta k_\beta l_\beta}(u)\cdots R^0_{i_\nu j_\nu k_\nu l_\nu}\Bigg).\end{aligned} \tag{3.22}$$

For the first term of the above equality we again use Lemma 3.3 and (3.6) to get

$$\nu\, A_W(n-1)\Big(-\frac{2n+1}{45}\sum_{a,b=1}^{n}R^2_{uaub}+\frac{1}{9}\rho^2_{uu}+\frac{1}{2}\nabla^2_{uu}\tau-\frac{n+3}{5}\nabla^2_{uu}\rho_{uu}\Big)(m). \tag{3.23}$$

Now, we briefly discuss the second term of (3.22), which is a simple directional curvature invariant of degree 4. Using the method of Lemma 3.3 to write the second addend of (3.22) as a linear combination of curvature invariants of degree 4 associated to $\widetilde{R}^2$ (see the basis (2.4)) we get expressions (3.14) and (3.15) . We delete the details. □

Remark 3.5. If $\nu=1$ in the previous theorem, we essentially have to deal with the scalar curvature $\widetilde{\tau}$. In this case, the second addend of (3.22) does not appear and $\omega_4(u)=0$. Then, $\widetilde{\tau}(\exp_m(ru))=\sum_{\alpha=-2}^{s-2}r^\alpha Sc(\widetilde{R}^{\alpha+2})+O\left(r^{s-1}\right)$. See [5] and [4] for an explicit power series expansion.

Example 3.6. The coefficients B_W^1 and B_W^2 in (3.14) for Weyl invariants of degree 4 and 6 can be given as follows:

W	$\|R\|^2$	$\|\rho\|^2$	τ^2
$B_W^1(n-1)$	1	0	0
$B_W^2(n-1)$	0	1	0

W	τ^3	$\tau\|\rho\|^2$	$\tau\|R\|^2$	$\check{\rho}$	$\langle\rho\otimes\rho,\bar{R}\rangle$	$\langle\rho,\dot{R}\rangle$	$\bar{R}$	$\check{\bar{R}}$
$B_W^1(n-1)$	0	0	$(n-1)(n-2)$	0	0	$n-2$	6	$-\frac{3}{2}$
$B_W^2(n-1)$	0	$(n-2)(n-1)$	0	$3(n-2)$	$2n-5$	4	0	3

Now, we derive some geometrical consequences of the expansions in Theorem 3.4. We recall that a manifold (M,g) is *Einstein* if $\rho=\frac{\tau}{n}\,g$. Moreover, an Einstein manifold is 2–*stein* if there exists a constant λ such that

$$\sum_{i,j=1}^{n} R_{xixj}^2=\lambda\, g(x,x)^2, \tag{3.24}$$

for all tangent vectors x.

Theorem 3.7. *Let (M^n,g) a Riemannian manifold and W a simple Weyl invariant of degree 2ν, $\nu>1$, such that*

$$\begin{aligned}
&A_W(n-1)\neq 0,\\
&(2n+1)\nu A_W(n-1)-20(n+12)B_W^1(n-1)+5n^2B_W^2(n-1)\neq 0,\\
&(2n+1)\nu A_W(n-1)+40nB_W^1(n-1)+5n^2B_W^2(n-1)\neq 0,
\end{aligned} \tag{3.25}$$

If the corresponding Weyl invariants of geodesic spheres $\widetilde{W}(\exp_m(ru))$ depend neither on the center m nor on the direction u then, M is 2–stein.

Proof. As $A_W(n-1)\neq 0$, using the coefficient $\widetilde{W}_2(u)$ given in Theorem 3.4, we get that $\tau-\frac{2(n+1)}{3}\rho_{uu}$ is independent of m and u. This implies that the manifold M is Einstein. Now, the coefficient $\widetilde{W}_4$ is also constant by hypothesis. Using the fact that M is Einstein, we obtain

$$\begin{aligned}
&B_W^1(n-1)\|R\|^2-4B_W^1(n-1)\sum_{a,b,c=1}^{n}R_{uabc}^2+\Big(-\frac{2n+1}{45}\nu A_W(n-1)\\
&\quad+\frac{4(n+12)}{9}B_W^1(n-1)+\frac{n^2}{9}B_W^2(n-1)\Big)\sum_{a,b=1}^{n}R_{uaub}^2=\text{constant}.
\end{aligned} \tag{3.26}$$

The above equation implies that the manifold is 2-stein if the last two conditions of (3.25) hold. For example, see [6, Lemma 4] for details. □

Remark 3.8. Let W be a simple Weyl invariant such that $A_W(n-1)\neq 0$. If M is a Riemannian manifold such that the corresponding Weyl invariants of geodesic spheres $\widetilde{W}(\exp_m(ru))$ depend only on the radius, then the above proof shows that M is an Einstein manifold.

Remark 3.9. It was proved in [4] that if $\widetilde{\tau}(\exp_m(ru))$ depends neither on the center nor on the direction u, then the manifold is 2-stein. Moreover, if the manifold is assumed to be analytic, then it is a harmonic space.

Example 3.10. It can easily be shown that conditions (3.25) hold for all the curvature invariants of Example 3.6. Hence, those may be used to characterize 2-stein manifolds.

Theorem 3.11. *Let (M, g) be an analytic Riemannian manifold with constant Weyl invariants and such that all its small geodesic spheres have constant scalar curvature. Then, M is locally isometric with a two-point homogeneous manifold or a Damek-Ricci space.*

Proof. As all the Weyl invariants of M are constant, M is locally homogeneous [15]. Also, as all the small geodesic spheres have constant scalar curvature and the manifold is analytic, M is harmonic [4]. Homogenous harmonic manifolds have been classified in [12]. According to this paper, M is locally isometric with a two-point homogeneous manifold or a Damek-Ricci space. □

Corollary 3.12. *Let (M, g) be an analytic Riemannian manifold with constant Weyl invariants such that all its small geodesic spheres also have constant Weyl invariants. Then, M is locally isometric to a two-point homogeneous space.*

Proof. Using the previous theorem, M is locally isometric to a two-point homogeneous space or a Damek-Ricci space. On the other hand, all the small geodesic spheres of M are homogenous, as they also have constant Weyl invariants. Hence, M is an Osserman space [8]. But a Damek-Ricci space cannot be an Osserman space unless it is symmetric [2]. The result follows because locally symmetric Damek-Ricci spaces are locally isometric to a two-point homogeneous space. □

4. Total scalar curvatures of geodesic spheres

Since a geodesic sphere is a compact Riemannian manifold one may consider the integral of the curvature invariant S for geodesic spheres. Following for example [4], we define the *total scalar curvature $\mathcal{S}$ associated to the scalar curvature invariant* S by

$$\mathcal{S}(m, r) = \int_{G_m(r)} \widetilde{S} = r^{n-1} \int_{S^{n-1}} (\widetilde{S}\, \theta_m)(\exp_m(ru))du, \tag{4.1}$$

where $\widetilde{S}$ is the corresponding curvature invariant of $G_m(r)$, θ_m is the volume density function at m and du is the volume element of S^{n-1}. Then, $\mathcal{S}$ is a function depending on the base point and the radius of the geodesic sphere.

Example 4.1. When (M, g) is a Riemannian manifold of constant sectional curvature $\lambda > 0$, each geodesic sphere $G_m(r)$ has constant sectional curvature $\widetilde{\lambda} = \frac{\lambda}{\sin^2 r\sqrt{\lambda}}$ (here, we only consider the positive curvature case; similar expressions can be obtained for negative and zero curvature). We now compute the total scalar curvature associated to a Weyl invariant W of degree 2ν. From Remark 2.1 we get

$$\widetilde{W} = (n-1)(n-2)A_W(n-1)\left(\frac{\lambda}{\sin^2 r\sqrt{\lambda}}\right)^{\nu}. \tag{4.2}$$

Furthermore, in a space of constant sectional curvature $\lambda > 0$ the volume density function is $\theta_m\left(\exp_m(ru)\right) = \left(\frac{\sin r\sqrt{\lambda}}{r\sqrt{\lambda}}\right)^{n-1}$ (see for example [10], [16]) and we have the exact expression for the total scalar curvature associated to W:

$$\int_{G_m(r)} \widetilde{W} = c_{n-1}(n-1)(n-2)A_W(n-1)\left(\frac{\sin r\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-1-2\nu}. \tag{4.3}$$

We emphasize that the above total scalar curvature does not depend on the base point m.

In order to obtain a power series expansion of a total scalar curvature we need the volume density function of a Riemannian manifold.

Lemma 4.2. *Let θ_m be the volume density function at a point m. Then we have*

$$\theta_m\left(\exp_m(ru)\right) = \sum_{\alpha=0}^{s} r^\alpha \theta_\alpha(u) + O\left(r^{s+1}\right), \tag{4.4}$$

where $\theta_\alpha(u)$, $\alpha \geq 2$ is a partial curvature invariant of degree α in the direction of u with α degrees of freedom. The first terms of this power series expansion are

$$\begin{aligned}\theta_m\left(\exp_m(ru)\right) =& 1 - \frac{1}{6}\rho_{uu}(m)r^2 - \frac{1}{12}\nabla_u\rho_{uu}(m)r^3 \\ &+\Big(-\frac{1}{180}\sum_{a,b=1}^{n} R^2_{uaub} + \frac{\rho^2_{uu}}{72} - \frac{1}{40}\nabla^2_{uu}\rho_{uu}\Big)(m)r^4 + O\left(r^5\right).\end{aligned} \tag{4.5}$$

Proof. We have the relation ([4], [16]), $h_m(\exp_m(ru)) = \frac{n-1}{r} + \frac{\partial}{\partial r}\log\theta_m\left(\exp_m(ru)\right)$, where h_m is the mean curvature of a geodesic sphere. Then,

$$\theta_\alpha(u) = \sum_{\beta=1}^{[\alpha/2]} \frac{1}{\beta!}\left(\sum_{\gamma_1+\cdots+\gamma_\beta=\alpha} \frac{h_{\gamma_1}(u)\cdots h_{\gamma_\beta}(u)}{\gamma_1\cdots\gamma_\beta}\right), \alpha \geq 2 \tag{4.6}$$

which proves the assertion concerning the degrees. The first terms of this power series expansion are well-known ([4], [16]). □

We define $c_{n-1} = n\pi^{\frac{n}{2}}/\left(\frac{n}{2}\right)!$, which is the volume of the Euclidean sphere of radius 1 in $\mathbb{R}^n$. Here $\left(\frac{n}{2}\right)! = \Gamma\left(\frac{n}{2}+1\right)$, where Γ is the gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt = \int_{-\infty}^{\infty} e^{-t^2}|t|^{2\alpha-1}dt$.

The following result is a technical lemma which will be used in the proof of the Theorem 4.4. We will only point out the main steps of the proof.

Lemma 4.3. *Let ω be a covariant tensor of order 2ν. Then,*

$$\int_{S^{n-1}} \omega_{u\cdots u}du = \frac{c_{n-1}}{2^\nu\nu!\prod_{\alpha=0}^{\nu-1}(n+2\alpha)} \sum_{\alpha_1\cdots\alpha_{2\nu}=1}^{n} \delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{2\nu-1}\alpha_{2\nu}} \sum_{\sigma\in S_{2\nu}} \omega_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(2\nu)}}. \tag{4.7}$$

Proof. We proceed by induction. If $\nu = 1$, (4.7) is a well-known fact. See, for example, [11]. Next, let ω be a covariant tensor of order $2(\nu+1)$. Choose an orthonormal basis $\{e_i\}$ at the origin of $\mathbb{R}^n$ and write the unit vector u with respect to that basis as $u = \sum_i x_i e_i$. Then

$$\int_{S^{n-1}} \omega_{u\cdots u} du = \sum_{\alpha_1\cdots\alpha_{2\nu+2}=1}^{n} \omega_{\alpha_1\cdots\alpha_{2\nu+2}} \int_{S^{n-1}} x_{\alpha_1}\cdots x_{\alpha_{2\nu+2}} du. \tag{4.8}$$

We recall the formula for integrating polynomials along Euclidean spheres [10]:

$$\int_{S^{n-1}} x_1^{\beta_1}\cdots x_n^{\beta_n} du = c_{n-1}\frac{\beta_1)\cdots\beta_n)}{n(n+2)\cdots(n+\beta_1+\cdots+\beta_n+2)}, \tag{4.9}$$

where

$$\begin{aligned} 2\beta) &= (2\beta-1)(2\beta-3)\cdots 3\cdot 1, \quad \beta\in\mathbb{N},\\ 0) = 1, &\qquad 2\beta-1) = 0, \quad \beta\in\mathbb{N}. \end{aligned} \tag{4.10}$$

Concentrating on the last index $\alpha_{2\nu+2}$ and using (4.9), (4.8) can be reduced to

$$\int_{S^{n-1}} \omega_{u\cdots u} du = \frac{1}{n+2\nu}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{2\nu+1}\left(\int_{S^{n-1}} \omega(u,\ldots,u,\underset{\underset{\beta}{\downarrow}}{e_\alpha},u,\ldots,u,e_\alpha)du\right). \tag{4.11}$$

Now the inner integral is a tensor of order 2ν, so we get (4.7) by induction. □

Theorem 4.4. *Let W be a simple Weyl invariant. The total scalar curvature associated to W has a power series expansion:*

$$\mathcal{W}(m,r) = \int_{G_m(r)} \widetilde{W} = c_{n-1}r^{n-1-2\nu}\left(\sum_{\alpha=0}^{[s/2]} r^{2\alpha}\frac{\mathcal{W}_{2\alpha}(m)}{\prod_{\beta=0}^{\alpha-1}(n+2\beta)} + O\left(r^{s+1}\right)\right), \tag{4.12}$$

where $\mathcal{W}_{2\alpha}(m)$, $\alpha\geq 1$ is a scalar curvature invariant of M at m of degree α, and

$$\begin{aligned} \mathcal{W}_0(m) &= (n-1)(n-2)A_W(n-1),\\ \mathcal{W}_2(m) &= \frac{(n-2)(n-2\nu-1)A_W(n-1)}{6}\tau(m),\\ \mathcal{W}_4(m) &= \Big(C_W^1(n-1)\|R\|^2 + C_W^2(n-1)\|\rho\|^2 + C_W^3(n-1)\tau^2\\ &\qquad -\frac{(n-2)(n-2\nu-1)A_W(n-1)}{20}\Delta\tau\Big)(m). \end{aligned} \tag{4.13}$$

Moreover, C_W^1, C_W^2 and C_W^3 are polynomials depending only on $n-1$ which are uniquely determined by (4.13). We also have the relation

$$\begin{aligned} 2C_W^1(n-1) + (n-1)C_W^2(n-1) + n(n-1)C_W^3(n-1) &=\\ = \frac{(n-2)(n+2)(n-1-2\nu)(5n-10\nu-7)}{360}A_W(n-1). \end{aligned} \tag{4.14}$$

Proof. By definition we have

$$\mathcal{W}(m,r) = \int_{G_m(r)} \widetilde{W} = r^{n-1}\int_{S^{n-1}}\left(\widetilde{W}\theta_m\right)(\exp_m(ru))du. \tag{4.15}$$

Using Theorem 3.4 and Lemma 4.2, we get

$$\left(\widetilde{W}\theta_m\right)(\exp_m(ru)) = \sum_{\alpha=-2\nu}^{s-2\nu} r^\alpha \bar{W}_{\alpha+2\nu}(u) + O\left(r^{s-2\nu+1}\right), \tag{4.16}$$

where $\bar{W}_\alpha(u)$, $\alpha \geq 2$ is a partial curvature invariant in the direction of u with degree α such that for all the Weyl invariants involved in its construction the number of degrees of freedom has the same parity as α. In fact, we have $\bar{W}_\alpha(u) = \sum_{\beta=0}^{\alpha} \widetilde{W}_\beta(u)\theta_{\alpha-\beta}(u)$, $\alpha \geq 2$, and in particular, using (3.13) and (4.5),

$$\begin{aligned}
\bar{W}_0(u) &= (n-1)(n-2)A_W(n-1), \\
\bar{W}_1(u) &= 0, \\
\bar{W}_2(u) &= A_W(n-1)\left(\nu\tau - \frac{4\nu(n+1)+(n-1)(n-2)}{6}\rho_{uu}\right)(m), \\
\bar{W}_4(u) &= \bar{\omega}_4(u) + \frac{A_W(n-1)}{2}\Bigg(\nu\nabla^2_{uu}\tau \\
&\quad -\frac{40\nu(n+3)+(n-1)(n-2)}{20}\nabla^2_{uu}\rho_{uu}\Bigg)(m),
\end{aligned} \tag{4.17}$$

where $\bar{\omega}_4(u)$ is a simple directional curvature invariant of degree 4. Then, (4.15) becomes

$$\mathcal{W}(m,r) = r^{n-1}\left(\sum_{\alpha=-2\nu}^{s-2\nu} r^\alpha \int_{S^{n-1}} \bar{W}_{\alpha+2\nu}(u)du + O\left(r^{s-2\nu+1}\right)\right). \tag{4.18}$$

If α is odd, $\bar{W}_{\alpha+2\nu}(u)$ is a linear combination of Weyl invariants in the direction of u with an odd number of degrees of freedom. Each one of them is an odd function on a sphere, and thus its integral vanishes. Hence, we have

$$\mathcal{W}(m,r) = r^{n-1}\left(\sum_{\alpha=-\nu}^{\left[\frac{s-2\nu}{2}\right]} r^{2\alpha} \int_{S^{n-1}} \bar{W}_{2\alpha+2\nu}(u)du + O\left(r^{s-2\nu+1}\right)\right). \tag{4.19}$$

The problem of integrating $\bar{W}_{2\alpha}(u)$, with $\alpha \geq 1$, reduces to the integration of directional Weyl invariants of degree 2α in the direction of u with an even number of degrees of freedom. This number is at most 2α. For such an invariant, Lemma 4.3 asserts that its integral is a linear combination of products of total traces, divided by certain polynomial. This immediately implies that $\int_{S^{n-1}} \bar{W}_{2\alpha+2\nu}(u)du$ is a curvature invariant at m with degree 2α, and (4.12) follows.

For the explicit expressions of $\bar{W}_0$, $\bar{W}_2$ and $\bar{W}_4$, we may use the general method described in this proof and just do the calculations taking into account (4.17). Examples of those may be found in [4], [10] and [11]. Finally, we integrate $\bar{\omega}_4(u)$. As this is a simple curvature invariant at m of degree 4 we have

$$\begin{aligned}
\int_{S^{n-1}} \bar{\omega}_4(u)du &= \frac{1}{n(n+2)}\big(C^1_W(n-1)\,\|R\|^2 \\
&\quad +C^2_W(n-1)\|\rho\|^2 + C^3_W(n-1)\tau^2\big)(m),
\end{aligned} \tag{4.20}$$

and from here we get the expression for $\mathcal{W}_4(m)$. For the Taylor power series expansion of the function in (4.3) we get

$$\begin{aligned}
\int_{G_m(r)} \widetilde{W} = c_{n-1}(n-1)(n-2)A_W(n-1)r^{n-1-2\nu}\Bigg(1 - \frac{n-1-2\nu}{6}\lambda r^2 \\
+\frac{(n-1-2\nu)(5n-10\nu-7)}{360}\lambda^2 r^4 + O\left(r^6\right)\Bigg).
\end{aligned} \tag{4.21}$$

Since for an n-dimensional space of constant sectional curvature λ we have $\tau = n(n-1)\lambda$, $\|R\|^2 = 2n(n-1)\lambda^2$, $\|\rho\|^2 = n(n-1)^2\lambda^2$, $\Delta\tau = 0$, (4.20) becomes

$$\begin{aligned} \int_{S^{n-1}} \bar{\omega}_4(u)du &= \frac{n-1}{n+2}\big(2C^1_W(n-1) \\ &\qquad +(n-1)C^2_W(n-1) + n(n-1)C^3_W(n-1)\big)\lambda^2. \end{aligned} \tag{4.22}$$

From the last two equations we get the desired result. □

We finish this section by writing down, for further use, the introduced functions B^1_W and B^2_W for the simple Weyl invariants of degree 2, 4 and 6 considered in Section 2.

First, for the scalar curvature we have

W	$C^1_W(n-1)$	$C^2_W(n-1)$
τ	$-\frac{(n+2)(n+3)}{120}$	$\frac{n^2+5n+21}{45}$

For degree 4 we have

W	$C^1_W(n-1)$	$C^2_W(n-1)$
$\|R\|^2$	$\frac{59n^2-93n-10}{60}$	$\frac{2(n^2-37n+60)}{45}$
$\|\rho\|^2$	$-\frac{n^3-9n^2-16n-20}{120}$	$\frac{n^3+31n^2-16n-120}{45}$
τ^2	$-\frac{(n-2)(n-1)(n^2+13n+10)}{120}$	$\frac{n^4+10n^3+43n^2-14n+120}{45}$

Finally, for order 6:

W	$C^1_W(n-1)$	$C^2_W(n-1)$
τ^3	$-\frac{(n-1)^2(n-2)^2(n^2+21n+14)}{120}$	$\frac{(n-1)(n-2)(n^4+18n^3+118n^2+105n+238)}{45}$
$\tau\|\rho\|^2$	$-\frac{(n-1)(n-2)(n^3-n^2-28n-28)}{120}$	$\frac{(n-2)(n^4+38n^3+28n^2+15n+238)}{45}$
$\tau\|R\|^2$	$\frac{(n-1)(n-2)(59n^2-101n-14)}{60}$	$\frac{2(n^4-32n^3+248n^2-135n+238)}{45}$
$\check{\rho}$	$-\frac{(n-2)(n^3-41n^2-28n-28)}{120}$	$\frac{(n-2)(n^3+79n^2-73n-238)}{45}$
$\langle\rho\otimes\rho,\bar{R}\rangle$	$-\frac{n^4-23n^3+34n^2+28n+56}{120}$	$\frac{n^4+57n^3-141n^2-2n+476}{45}$
$\langle\rho,\dot{R}\rangle$	$\frac{59n^3-179n^2+188n+28}{60}$	$\frac{2(n^3+9n^2+77n-238)}{45}$
$\bar{R}$	$\frac{179n^2-261n-14}{30}$	$\frac{4(n^2-129n+119)}{45}$
$\check{\bar{R}}$	$-\frac{n^3+138n^2-289n-42}{120}$	$\frac{n^3+78n^2+56n-357}{45}$

5. Homogeneity and two-point homogeneous spaces

If M is a locally homogeneous Riemannian manifold, its total scalar curvatures $\mathcal{W}(m,r) = \int_{G_m(r)} \widetilde{W}$ do not depend on the point m and thus one may wonder whether the converse is also true. The answer is known to be positive for several special cases, but the problem remains open in its full generality. In our general context, positive answers can be given in a similar way as a consequence of Theorem 4.4 and the following result (we omit the details, which are similar to those in [3]).

Theorem 5.1. *Let W be a Weyl invariant such that $A_W(n-1) \neq 0$. If a Riemannian manifold (M^n, g) of dimension $n > 2, n \neq 2\nu + 1$ verifies that $\mathcal{W}(m, r)$ is independent of the point m, then the scalar curvature and the quadratic invariant $C^1_W(n-1)\|R\|^2 + C^2_W(n-1)\|\rho\|^2$ are constant.*

In particular, if $W = 1$, $\mathcal{S}(m, r) = \int_{G_m(r)} 1$ is the volume of the geodesic sphere $G_m(r)$. A manifold having the property that the volume of geodesic spheres is independent of the center is called *ball-homogeneous* [3], [7]. Also, a Riemannian manifold is said to be *scalar curvature homogeneous* if $\mathcal{T}(m, r) = \int_{G_m(r)} \widetilde{\tau}$ is independent of m [3], [9] (this is also a particular case of our context for $W = \tau$). Next, we show that both notions above are equivalent for Einstein manifolds, thus answering the question already stated in [3].

Theorem 5.2. *Ball-homogeneity and scalar curvature homogeneity are equivalent in the class of Einstein manifolds.*

Proof. We denote by $'$ the derivative with respect to the radius r. We recall the relation [16]:

$$h_m(\exp_m(ru)) = \frac{n-1}{r} + \frac{\partial}{\partial r} \log \theta_m(\exp_m(ru)), \tag{5.1}$$

where h_m denotes the mean curvature of the geodesic sphere. Deriving, and using (5.1), we get

$$S'(m, r) = \frac{d}{dr}\left[r^{n-1}\int_{S^{n-1}} \theta_m du\right] = r^{n-1}\int_{S^{n-1}} (h_m\, \theta_m) du. \tag{5.2}$$

Again, deriving with respect to the radius and using (5.1), we obtain

$$\begin{aligned} S''(m, r) &= \frac{d}{dr}\left[r^{n-1}\int_{S^{n-1}} (h_m\, \theta_m) du\right] \\ &= r^{n-1}\int_{S^{n-1}} \left(\frac{n-1}{r}\, h_m\, \theta_m + h'_m\, \theta_m + h_m\, \theta'_m\right) du \\ &= r^{n-1}\int_{S^{n-1}} \left(\left(h_m^2 + h'_m\right) \theta_m\right) du. \end{aligned} \tag{5.3}$$

Taking traces in the Gauss equation, we get the scalar curvature of a geodesic sphere $G_m(r)$

$$\widetilde{\tau} = \tau - 2\,\rho_{uu} + h_m^2 - \|\sigma_m\|^2. \tag{5.4}$$

Next, we consider the Ricatti equation, $\sigma' + \sigma^2 + R_u = 0$, where R_u is the Jacobi operator $x \mapsto R_{ux}u$. Taking traces, we get $h'_m + \|\sigma_m\|^2 + \rho_{uu} = 0$. Therefore, (5.4) becomes

$$\widetilde{\tau} = \tau - \rho_{uu} + h_m^2 + h'_m. \tag{5.5}$$

Using the above equality, (5.3) becomes

$$S''(m, r) = \mathcal{T}(m, r) - r^{n-1}\int_{S^{n-1}} ((\tau - \rho_{uu})\, \theta_m). \tag{5.6}$$

In the class of Einstein manifolds $\tau - \rho_{uu} = \frac{n-1}{n}\, \tau$ is constant. Thus, we get

$$\mathcal{S}''(m, r) = \mathcal{T}(m, r) - \frac{n-1}{n}\, \tau\, \mathcal{S}(m, r). \tag{5.7}$$

As τ is constant, $\mathcal{T}$ depends on m if and only if $\mathcal{S}$ depends on m. $\square$

Now, we turn our attention to the characterization of two-point homogeneous spaces using the total curvatures of geodesic spheres associated to simple Weyl invariants.

Theorem 5.3. *Let (M^n, g) be a Riemannian manifold of dimension $n > 2$. Suppose that the total scalar curvature associated to a simple Weyl invariant of degree ν is the same as for a Riemannian manifold of constant sectional curvature λ. If*

$$\begin{aligned} &A_W(n-1) \neq 0, \quad n \neq 2\nu + 1,\\ &C_W^1(n-1) \neq 0,\\ &C_W^1(n-1)\left(C_W^2(n-1) + \frac{2}{n-1}C_W^1(n-1)\right) \geq 0, \end{aligned} \tag{5.8}$$

then M is a Riemannian manifold of constant sectional curvature λ.

Proof. As we have already seen that for a manifold of constant sectional curvature λ, we have

$$\begin{aligned} \int_{G_m(r)} \widetilde{W} = c_{n-1}(n-1)(n-2)A_W(n-1)r^{n-1-2\nu}\Bigg(1 - \frac{n-1-2\nu}{6}\lambda\, r^2 \\ + \frac{(n-1-2\nu)(5n-10\nu-7)}{360}\lambda^2 r^4 + O\left(r^6\right)\Bigg). \end{aligned} \tag{5.9}$$

As $A_W(n-1) \neq 0$ and $n \neq 2\nu+1$, comparing (5.9) with (4.12) and (4.13) we immediately get $\tau = n(n-1)\lambda$. Then, (4.12) becomes

$$\begin{aligned} \mathcal{W}(m,r) \;=\; & c_{n-1}r^{n-1-2\nu}\Big\{(n-1)(n-2)A_W(n-1)\\ & -\frac{r^2}{6}(n-2)(n-1-2\nu)A_W(n-1)(n-1)\lambda\\ & +\frac{r^4}{n(n+2)}\Big(C_W^1(n-1)\|R\|^2 + C_W^2(n-1)\|\rho\|^2\\ & \qquad + n^2(n-1)^2C_W^3(n-1)\lambda^2\Big)(m) + O\left(r^6\right)\Big\}(m). \end{aligned} \tag{5.10}$$

Comparing the coefficients of r^4 in (5.9) and (5.10) and taking into account that $\tau = n(n-1)\lambda$ we easily get

$$\begin{aligned} & C_W^1(n-1)\left(\|R\|^2 - \frac{2}{n-1}\|\rho\|^2\right)\\ & \quad + \left(C_W^2(n-1) + \frac{2}{n-1}C_W^1(n-1)\right)\left(\|\rho\|^2 - \frac{1}{n}\tau^2\right) = 0. \end{aligned} \tag{5.11}$$

We know that for every Riemannian manifold of dimension $n > 2$, $\|\rho\|^2 \geq \frac{1}{n}\tau^2$ with equality if and only if the manifold is Einstein and $\|R\|^2 \geq \frac{2}{n-1}\|\rho\|^2$ with equality if and only if the manifold has constant sectional curvature [1]. The hypotheses of this theorem imply that both terms of the left-hand side of (5.11) are simultaneously non-negative or non-positive (depending on the sign of $C_W^1(n)$). Then, both addends must be zero and hence $\|R\|^2 = \frac{2}{n-1}\|\rho\|^2$. Thus, M has constant sectional curvature. As $\tau = n(n-1)\lambda$, the sectional curvature is precisely λ. $\square$

We state similar theorems for the other two-point homogeneous spaces. See [4] or [11] for more information. We delete the details.

Theorem 5.4. *Let (M^{2n}, g, J) be a Kähler manifold of complex dimension $n > 1$. Suppose that the total scalar curvature associated to a simple Weyl invariant of degree ν is the same as for a Kähler manifold of constant holomorphic sectional curvature λ. If*

$$\begin{aligned} &A_W(2n-1) \neq 0, \\ &C^1_W(2n-1) \neq 0, \\ &C^1_W(2n-1)\left(C^2_W(2n-1) + \frac{4}{n+1} C^1_W(2n-1)\right) \geq 0, \end{aligned} \tag{5.12}$$

then M is a Kähler manifold of constant holomorphic sectional curvature λ.

Theorem 5.5. *Let M^{4n} be a quaternionic Kähler manifold of real dimension $4n$. Suppose that the total scalar curvature associated to a simple Weyl invariant of degree ν is the same as for a quaternionic Kähler manifold of constant Q-sectional curvature λ. If*

$$A_W(4n-1) \neq 0, \qquad C^1_W(4n-1) \neq 0, \tag{5.13}$$

then M is a quaternionic Kähler manifold of constant Q-sectional curvature λ.

Combining the previous results we get:

Theorem 5.6. *Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$ such that its holonomy group is contained in the holonomy group of a two-point homogeneous space of dimension n. Suppose that the total scalar curvature associated to a simple Weyl invariant of degree ν is the same as for the corresponding two-point homogeneous space. If*

$$\begin{aligned} &A_W(n-1) \neq 0, \quad n \neq 2\nu + 1, \\ &C^1_W(n-1) \neq 0, \\ &C^1_W(n-1)\left(C^2_W(n-1) + \frac{2}{n-1} C^1_W(n-1)\right) \geq 0, \end{aligned} \tag{5.14}$$

then M is locally isometric to that space.

Example 5.7. Using the expressions of examples 2.2 and 3.6, we may check the conditions of Theorem 5.6. We give a table with those simple Weyl invariants which can be used for characterizing the two-point homogeneous spaces and the dimension n for which the conditions of Theorem 5.6 hold.

W	$C^2_W(n-1) + \frac{2}{n-1} C^1_W(n-1)$	n
$\|R\|^2$	$\frac{4n^3+25n^2+109n-270}{90(n-1)}$	$n > 2$, $n \neq 5$
$\|\rho\|^2$	$\frac{4n^4+117n^3-161n^2-368n+540}{180(n-1)}$	$3 \leq n \leq 10$, $n \neq 5$
$\tau\|\rho\|^2$	$\frac{(n-2)(4n^4+149n^3+115n^2+144n+1036)}{180}$	$3 \leq n \leq 6$
$\tau\|R\|^2$	$\frac{4n^4+49n^3+335n^2+24n+1036}{90}$	$n > 2$, $n \neq 7$
$\check{\rho}$	$\frac{(n-2)(4n^4+309n^3-485n^2-576n+1036}{180(n-1)}$	$3 \leq n \leq 41$, $n \neq 7$
$\langle \rho \otimes \rho, \bar{R} \rangle$	$\frac{4n^5+221n^4-723n^3+454n^2+1828n-2072}{180(n-1)}$	$3 \leq n \leq 21$, $n \neq 7$
$\langle \rho, \dot{R} \rangle$	$\frac{4n^4+209n^3-265n^2-696n+1036}{90(n-1)}$	$n > 2$, $n \neq 7$
$\check{R}$	$\frac{(n-2)(4n^2+25n+259)}{45(n-1)}$	$n > 2$, $n \neq 7$

Remark 5.8. If $n = 3$, the Gauss-Bonnet Theorem gives $\mathcal{T}(m, r) = 8\pi$. Hence, $\mathcal{T}$ is a topological invariant. Generalizations of the Gauss-Bonnet Theorem show that some total scalar curvatures have no geometrical meaning in certain dimensions [5].

Let now W be a simple Weyl invariant of order 2ν. Consider a Riemannian manifold of constant sectional curvature of dimension $2\nu + 1$. Then (4.3) shows

$$\mathcal{W}(m, r) = 2\nu(2\nu - 1)c_{2\nu}A_W(2\nu). \tag{5.15}$$

Thus, $\mathcal{W}(m, r)$ is an invariant for $(2\nu + 1)$-dimensional manifolds of constant sectional curvature and therefore it cannot be used to determine the curvature.

Remark 5.9. The third condition in Theorem 5.6 can be dropped if the manifold is assumed to be Einstein or locally conformally flat (see [5] or [11] for similar situations).

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